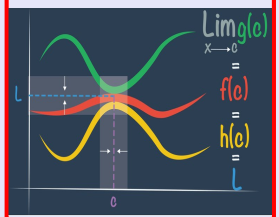
Calculus I

Lecture 13



Feb 19-8:47 AM

Class QZ 10

Given $\boxed{x^3} + \boxed{y^3} = 6xy$

1) find $\frac{dy}{dx}$

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

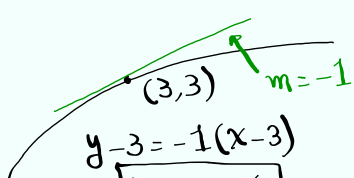
$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)}$$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

2) Evaluate $\left.\frac{dy}{dx}\right|_{(3,3)}$

$$\left.\frac{dy}{dx}\right|_{(3,3)} = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = \frac{6-9}{9-6} = \boxed{-1}$$



$$y - 3 = -1(x - 3)$$

$$\boxed{y = -x + 6}$$

Jul 8-7:11 AM

At what point the graph of $x^3+y^3=6xy$ has horizontal Tan. line?

$m=0$

$\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$

$2y-x^2=0$ as long as $y^2-2x\neq 0$

Solve $\begin{cases} 2y-x^2=0 \rightarrow y=\frac{x^2}{2} \\ x^3+y^3=6xy \end{cases}$

$x^3+\left(\frac{x^2}{2}\right)^3=\cancel{6}^{3}x\left(\frac{x^2}{\cancel{2}}\right)$

$x^3+\frac{x^6}{8}=3x^3$

$8x^3+x^6=24x^3$

$x^6+8x^3-24x^3=0$

$x^6-16x^3=0$

$x^3(x^3-16)=0$

$x^3=0$ $x^3-16=0$

$x=0$ $x=\sqrt[3]{16}$

$x=\sqrt[3]{8}\sqrt[3]{2}$

$x=2\sqrt[3]{2}$

$x=0$

$y=\frac{x^2}{2}=0$

$y^2-2x\neq 0$

$0^2-2(0)\neq 0$

$0\neq 0$

false

$(0,0)$ is not the point.

$x=2\sqrt[3]{2}$

$y=\frac{(2\sqrt[3]{2})^2}{2}$

$=\frac{4\sqrt[3]{4}}{2}$

$y=2\sqrt[3]{4}$

$y^2-2x\neq 0$

$(2\sqrt[3]{4})^2-2\cdot 2\sqrt[3]{2}\neq 0$

$4\sqrt[3]{16}-4\sqrt[3]{2}\neq 0$

$4\sqrt[3]{8}\sqrt[3]{2}-4\sqrt[3]{2}\neq 0$

$8\sqrt[3]{2}-4\sqrt[3]{2}\neq 0$

$4\sqrt[3]{2}\neq 0$

$(2\sqrt[3]{2}, 2\sqrt[3]{4})$ is the point where tangent line is horizontal.

Jul 8-8:20 AM

$x^2+y^2=16$

1) find y' $2x+2yy'=0$ $y'=\frac{-2x}{2y}$

$\boxed{y'=\frac{-x}{y}}$

2) find y''

$y''=\frac{d}{dx}[y']=\frac{d}{dx}\left[\frac{-x}{y}\right]$

$=-\frac{d}{dx}\left[\frac{x}{y}\right]=-\frac{1\cdot y-x\cdot y'}{y^2}$

$=-\frac{y-x\cdot\frac{-x}{y}}{y^2}$

$=-\frac{y+\frac{x^2}{y}}{y^2}=-\frac{y^2+x^2}{y^3}$

$\boxed{y''=\frac{-16}{y^3}}$

Jul 8-8:38 AM

$g(x) + x \sin g(x) = x^2$ Find $g'(0)$.

$f(x) + x \sin f(x) = x^2$ find $f'(0)$

$y + x \sin y = x^2$ find $\frac{dy}{dx}$ at $x=0$

$y + x \sin y = x^2$

$y + \cancel{0 \sin y}^{0} = 0^2$

$y=0$

$\frac{dy}{dx} + 1 \sin y + x \cdot \cos y \cdot \frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{(0,0)}$

$(1 + x\cos y)\frac{dy}{dx} = 2x - \sin y$

$\frac{dy}{dx} = \frac{2x - \sin y}{1 + x\cos y}$

$\left.\frac{dy}{dx}\right|_{(0,0)} = \frac{2 \cdot 0 - \cancel{\sin 0}^{0}}{1 + 0 \cdot \cancel{\cos 0}^{1}}$

$= \frac{0}{1} = \boxed{0}$

Jul 8-8:47 AM

$x^2 + xy + y^3 = 1$

1) find y when $x = 1$

$(1, 0)$

$\cancel{1^2} + 1y + y^3 = \cancel{1}$

$y^3 + y = 0$

$y(y^2 + 1) = 0$

$y = 0$

y^2+1 → has no real Soln.

2) find y', y'', and y'''

$2x + 1 \cdot y + xy' + 3y^2 y' = 0$

$(x + 3y^2)y' = -2x - y$

3) Evaluate y''' at $x=1$.

$\boxed{y' = \frac{-2x - y}{x + 3y^2}}$

$y'' = \frac{(-2 - y')(x + 3y^2) - (-2x - y)(1 + 6yy')}{(x + 3y^2)^2}$

$y'' = \frac{\cancel{-2x} - 6y^2 - xy' - \cancel{3y^2 y'} + \cancel{2x} + 12xyy' + y + \cancel{6y^2 y'}^{+3y^2y'}}{(x + 3y^2)^2}$

$y'' = \frac{-6y^2 - xy' + 12xyy' + 3y^2 y'}{(x + 3y^2)^2}$

$y'' = \frac{-6y^2 - (x - 12xy - 3y^2)y'}{(x + 3y^2)^2}$

$= \frac{-6y^2 - (x - 12xy - 3y^2)\frac{-2x - y}{x + 3y^2}}{(x + 3y^2)^2} = \frac{-6y^2(x + 3y^2) - (x - 12xy - 3y^2) \cdot (-2x - y)}{(x + 3y^2)^3}$

$(x - 12xy - 3y^2)(2x + y)$

$= \frac{-6xy^2 - 18y^4 + 2x^2 + xy - 24x^2y - 12xy^2 - 6xy^2 - 3y^3}{(x + 3y^2)^3}$

$= \frac{-24xy^2 - 18y^4 + 2x^2 + xy - 24x^2y - 3y^3}{(x + 3y^2)^3}$

Jul 8-8:58 AM

Related Rates

If $x = x(t)$, $y = y(t)$

time

we take derivative with respect to t.

$$x^2 + y^3 = 100$$

$\frac{dx}{dt}$, $\frac{dy}{dt}$

changes in x & y with respect to time

$$\frac{d}{dt}\left[x^2 + y^3\right] = \frac{d}{dt}\left[100\right]$$

$$2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

Suppose $x = 2$, $\frac{dx}{dt} = 5$, and $y = 3$ → find $\frac{dy}{dt}$

$$2 \cdot 2 \cdot 5 + 3 \cdot 3^2 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-20}{27}$$

$\frac{dx}{dt} > 0$ $\qquad$ $\frac{dy}{dt} < 0$

x increasing $\qquad$ y decreasing

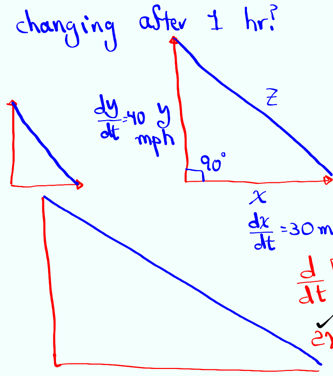
Jul 8-9:40 AM

Two Cars leave an intersection at the Same time

East → 30 mph

North → 40 mph

How fast is the distance between them changing after 1 hr?



$$x^2 + y^2 = z^2$$

In 1 hr

$x = 30$, $y = 40$

$$30^2 + 40^2 = z^2$$

$$z = 50$$

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}\left[z^2\right]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$30 \cdot 30 + 40 \cdot 40 = 50\frac{dz}{dt}$$

$$2500 = 50\frac{dz}{dt}$$

$$\boxed{\frac{dz}{dt} = 50 \text{ mph}}$$

Jul 8-9:47 AM

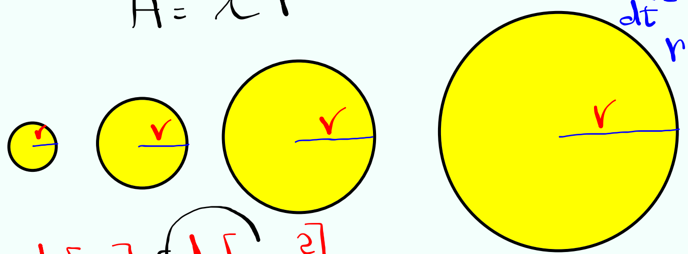
Fire is spreading in a circular form. How fast its area changes when radius is 4 ft and radius is increasing at the rate of 2 ft/min?

$\frac{dr}{dt} = 2$ ft/min

$\frac{dA}{dt} = ?$

$r = 4$ ft

$A = \pi r^2$



$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$

$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi \cdot 4 \cdot 2$

$= 16\pi$ ft^2/min.

Jul 8-9:55 AM

A basketball is losing air at the rate of 5 cm^3/min.

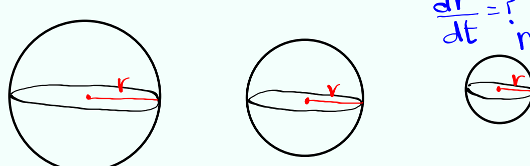
$\frac{dV}{dt} = -5$ cm^3/min.

How fast is its radius decreasing when the radius is 10 cm?

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = ?$

$r = 10$ cm



$\frac{d}{dt}[V] = \frac{d}{dt}[\frac{4}{3}\pi r^3]$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-5 = 4\pi(10)^2 \frac{dr}{dt}$

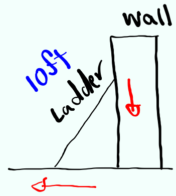
$-5 = 400\pi \frac{dr}{dt}$

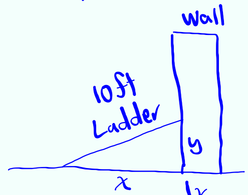
$\frac{dr}{dt} = \frac{-5}{400\pi}$ cm/min

$\boxed{\frac{dr}{dt} = \frac{-1}{80\pi} \text{ cm/min}}$

Jul 8-10:01 AM

A 10-ft ladder is leaning against a wall. the bottom of the ladder is pulled away from the wall at 4 ft/min. Top of the ladder is Coming down. How fast is the top dropping when bottom is 6 ft & Top is 8 ft?





$\frac{dy}{dt} = -?$

$\frac{dx}{dt} = 4$

when $x=6$, $y=8$

$x^2 + y^2 = 10^2$

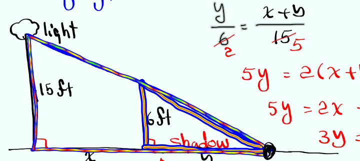
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$6 \cdot 4 + 8\frac{dy}{dt} = 0 \longrightarrow \frac{dy}{dt} = \frac{-24}{8}$

$= -3$ ft/min.

Jul 8-10:10 AM

Street light is 15-ft tall. A man is 6 ft tall walks away from the light at $\underbrace{\text{5 ft/s along}}_{\frac{dx}{dt}=5}$ a Straight Path. How fast is the $\underbrace{\text{tip of his shadow}}$ changing?



$\frac{y}{\cancel{6}\,2} = \frac{x+y}{\cancel{15}\,5}$

$5y = 2(x+y)$

$5y = 2x + 2y$

$3y = 2x$

length of shadow

$3\frac{dy}{dt} = 2\frac{dx}{dt}$

$3\frac{dy}{dt} = 2 \cdot 5$

$\frac{dy}{dt} = \frac{10}{3}$ ft/Sec.

length of shadow

Answer is different

Rate of Tip of shadow

$\frac{d}{dt}[x+y] = \frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{10}{3} = \frac{25}{3}$ ft/Sec.

Jul 8-10:20 AM

Estimate $\cos 59^\circ$

from Calculator $\longrightarrow \approx .515$ ✓

Common Sense

$\cos 59^\circ \approx \cos 60^\circ$

Linear Approximation

$f(x) = \cos x$

$a = 60^\circ$

$\cos 60^\circ = \frac{1}{2}$

$f'(x) = -\sin x$

$f'(60^\circ) = -\sin 60^\circ$

$= -\frac{\sqrt{3}}{2}$

$180^\circ = \pi$ Rad.

$1^\circ = \frac{\pi}{180}$ Rad

$f(x) \approx f(a) + f'(a)(x-a)$

$\cos x \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$

$\cos 59^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}(59^\circ - 60^\circ)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}(-1^\circ)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot 1^\circ$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$

$= \frac{1}{2} + \frac{\pi\sqrt{3}}{360}$

$\approx .515$ ✓

Jul 8-10:33 AM

$f(x) = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x$

1) Find $f'(x)$

$f'(x) = \frac{1}{5} \cdot 5x^4 - \frac{8}{3} \cdot 3x^2 + 16$

$f'(x) = x^4 - 8x^2 + 16$

2) Solve $f'(x) = 0$

$x^4 - 8x^2 + 16 = 0$

$(x^2 - 4)^2 = 0$

$x^2 - 4 = 0$

$x^2 = 4 \longrightarrow \boxed{x = \pm 2}$

3) Find $f''(x)$

$f''(x) = 4x^3 - 16x$

4) Solve $f''(x) = 0$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$\boxed{x = 0}$ $\quad \boxed{x = \pm 2}$

Jul 8-10:41 AM

$$f(x) = \frac{x^2-4}{x^2-2x} \qquad \text{Find } f'(x) \text{ \& } f''(x)$$

$$f(x) = \frac{(x+2)\cancel{(x-2)}}{x\cancel{(x-2)}} = \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + 2x^{-1}$$

$$f(x) = 1 + 2x^{-1}$$

$$\boxed{f'(x) = -2x^{-2}} \qquad \boxed{f''(x) = 4x^{-3}}$$

$$f'(x) = \frac{-2}{x^2} \qquad f''(x) = \frac{4}{x^3}$$

Jul 8-10:49 AM

$$f(x) = \frac{x^2}{x^2+9} = \frac{x^2+9-9}{x^2+9} = \frac{x^2+9}{x^2+9} - \frac{9}{x^2+9}$$

Find $f'(x)$ & $f''(x)$ $\qquad = 1 - 9(x^2+9)^{-1}$

$$f'(x) = -9\cdot(-1)(x^2+9)^{-2}\cdot 2x = \boxed{\frac{18x}{(x^2+9)^2}}$$

$$f'(x) = 18x(x^2+9)^{-2}$$

$$f''(x) = 18\left[1(x^2+9)^{-2} + x\cdot -2(x^2+9)^{-3}\cdot 2x\right]$$

$$= 18\left[1(x^2+9)^{-2} - 4x^2(x^2+9)^{-3}\right]$$

$$= 18(x^2+9)^{-3}\left[1(x^2+9)^{1} - 4x^2\right]$$

$$= \frac{18(9-3x^2)}{(x^2+9)^3} = \boxed{\frac{54(3-x^2)}{(x^2+9)^3}}$$

Jul 8-10:55 AM